THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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THE TEACHING OF EUCLID.

It has been customary when Euclid, considered as a text-book, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test.

The first proposition assumes that the circles used in the construction intersect—an assumption not noticed by Euclid because of the dangerous habit of using a figure. We require as a lemma, before the construction can be known to succeed, the following: If A and B be any two given points, there is at least one point C whose distances from A and B are both equal to AB. This lemma may be derived from an axiom of continuity. The fact that in elliptic space it is not always possible to construct an equilateral triangle on a given base, shows also that Euclid has assumed the straight line to be not a closed curve—an assumption which certainly is not made explicit. When these facts are taken account of, it will be found that the first proposition has a rather long proof, and presupposes the fourth. We require the axiom: on any straight line there is at least one point whose distance from a given point on or off the line exceeds a given distance.

The fourth proposition is a tissue of nonsense. Superposition is a logically worthless device; for if our triangles are spatial, not material, there is a logical contradiction in the notion of

moving them, while if they are material, they cannot be perfectly rigid, and when superposed they are certain to be slightly deformed from the shape they had before. What is presupposed, if anything analogous to Euclid's proof is to be retained, is the following very complicated axiom: Given a triangle ABC and a straight line DE, there are two triangles, one on either side of DE, having their vertices at D, and one side along DE, and equal in all respects to the triangle ABC. (This axiom presupposes the definition of the two sides of a line, for which see below.) When the existence of a triangle thus equal in all respects to ABC is assured, we can prove that the triangle considered in the fourth proposition is this triangle.

The sixth proposition requires an axiom which may be stated as follows: If OAA', OBB', OCC' be three lines in a plane, meeting two transversals in A, B, C, A', B', C' respectively; and if O be not between A and A', nor B and B', nor C and C', or be between in all three cases; then, if B be between A and C, B' is between A' and C'. This axiom is the basis of the measurement of angles by distances, and is required for proving that if D be on AB, and BD be less than BA, the triangle DBC is less than the triangle ABC.

The seventh proposition is so thoroughly fallacious that Euclid would have done better not to attempt a proof. In the first place, it uses an undefined term in the enunciation, namely, *on the same side*. The definition requires an axiom, and may be set forth as follows: Given a line AB and a point C, with regard to any point D in the plane ABC, three cases may arise; (1) the straight line CD does not meet AB; (2) CD meets AB, produced if necessary, in a point not between C and D; (3) CD meets AB in a point between C and D. In cases (1) and (2), C and D are said to be on the same side of AB; in case (3), on opposite sides. The above very complicated axiom is better replaced by the following two: (1) Given three points A, B, C, a point D between B and C, and a point G between A and D, BG produced meets AC in a point between A and C; (2) A, B, C, D being as before, and E being between A and C, AD and BE meet in a point between A and D and also between B and E.* (The definition of *between* is long, and I omit it here for want of space.) The proof of I. 7 further assumes that if C and D be on the same side of AB, then if CB is between CA and CD, DA is between DC and DB; while if CB is between CD and AC produced, then AD produced is between DC and DB. This is a very complicated assumption, of which Euclid is to all appearance completely ignorant. The assumption may be stated more simply as follows: Of three lines in a plane starting from a point, either there is one which is

* Cf. Pasch, *Vorlesungen über neuere Geometrie*, Leipzig, 1882; Peano, *I Principii di Geometria*, Turin, 1889.

between the other two, or else any one of them produced is between the other two. But in this statement, the meaning of *between* has to be very carefully defined.

I. 8 involves the same fallacy as I. 4, and requires the same axiom as to the existence of congruent triangles in different places. In the following propositions, we require the equality of all right angles, which is not a true axiom, since it is demonstrable.* I. 12 involves the assumption that a circle meets a line in two points or in none, which has not been in any way demonstrated. Its demonstration requires an axiom of continuity, by the help of which the circle can be dispensed with as an independent figure.

I. 16 is false in elliptic space, although Euclid does not explicitly employ any assumption which fails for that space. Implicitly, he uses the following: If ABC be a triangle, and E the middle point of AC; and if BE be produced to F so that $BE=EF$, then CF is between CA and BC produced. In spaces where the straight line is not a closed series, this follows from the axioms mentioned in connection with I. 6 and I. 7. No other points of interest, except that I. 26 involves the same fallacy as I. 4 and I. 8, arise until we come to parallels; and the treatment of parallels in Euclid is, so far as I know, wholly free from logical defects.

Many more general criticisms might be passed on Euclid's methods, and on his conception of Geometry; but the above definite fallacies seem sufficient to show that the value of his work as a masterpiece of logic has been very grossly exaggerated.

B. RUSSELL.

THE COMMITTEE ON GEOMETRY.

THE following report, preliminary and subject to revision, has been drawn up by a Committee of the Mathematical Association, consisting of representatives from a large number of public schools, especially of those near London. It is the outcome of many meetings, and of prolonged deliberation on the more drastic of the changes proposed. It affords striking evidence, if any were needed, of the fact that mathematical teachers are neither unconscious of, nor indifferent to, the condition of things so forcibly depicted at the last meeting of the British Association. It is gratifying to note, *en passant*, that the counsels of the Committee were on the whole pervaded by singular unanimity.

Similar reports will shortly be issued on the teaching of Arithmetic and Algebra. The reports taken as a whole will represent a body of opinion which cannot be ignored, and should have a wholesome effect upon the future of mathematical teaching in this country.

It is hoped that the readers of the *Gazette* will make a serious study of the reports. All criticisms and suggestions will receive careful consideration at the hands of the Committee. They should, in the first instance, be sent to the Secretary, Mr. A. W. Siddons, Harrow School, Middlesex. It is very desirable that mathematical masters and others should fully avail themselves of this opportunity of placing on record their views as to the proposed changes; and, it is hardly necessary to add, that the hands of the Committee

* Cf. Hilbert, *Grundlagen der Geometrie*, Leipzig, 1899, p. 16.

will be greatly strengthened by finding that the mass of their colleagues in this profession endorse the proposals.

It has long been felt that changes were inevitable, and we hope that the suggestions made by the Committee will be approved by those who have given time and thought to the question.

Res ipsa loquitur, judices, quae semper valet plurimum. [W. J. G.]

REPORT OF THE M.A. COMMITTEE ON GEOMETRY.

THE Committee make the following suggestions under the head of Geometry.

Introductory and Experimental Course.

It is desirable

1. That a first introduction to Geometry should not be formal but experimental, with use of instruments and numerical measurements and calculations.

2. That Public Schools in their entrance examinations should set a fair proportion of questions requiring the use of instruments, and the obtaining of numerical results from numerical data by measurements from accurately drawn figures; and that in their entrance scholarship examinations the same principle should be recognised.

3. That elementary geometry papers, in examinations such as University Local Examinations, the Examinations of the College of Preceptors, Oxford Responsions and the Cambridge Previous Examination, should contain some questions requiring the practical use of instruments.

Division of the formal course into two parallel courses of (i) *Theorems,* (ii) *Constructions.*

4. Since pupils will have been already familiarised with the principal constructions of Euclid before they begin their study of formal geometry, it is desirable that the course of constructions should be regarded as quite distinct from the course of theorems. The two courses will probably be studied side by side, but great freedom should be allowed to the teacher as to the order in which he takes the different constructions.

With regard to Constructions.

5. The course of constructions should be regarded as a *practical* course, the constructions being accurately made with instruments, and no construction, or proof of a construction, should be deemed invalid by reason of its being different from that given in Euclid, or by reason of its being based on theorems which Euclid placed after it.

With regard to Theorems.

6. The Committee propose, with a view to making the course of theorems independent of methods of construction, that no proof of a theorem should be considered invalid by reason of an assumption that a line or angle may be divided into any number of equal parts or that a line may be drawn from any point in any assigned direction and of any assigned length, or that any figure may be duplicated or placed in any position.

7. It is not proposed to interfere with the logical order of Euclid's series of theorems;—in other words, it is not proposed to introduce any order of theorems that would render invalid Euclid's proof of any proposition.

8. As far as possible, proofs of theorems should be based on first principles, and long chains of dependent propositions should be avoided.

9. Proof of *congruence* by superposition, and, in particular, proof of symmetry about a line by folding should be considered fundamental methods of proof.

10. Connected theorems should be associated together in the pupil's mind. *e.g.* (I. 13, 14, 15), (I. 4, 8, 26), (VI. 4, 5, 6, 7), and, in particular, a theorem and its converse, when true, should always be so associated.

The importance of Riders.

11. In pass examinations it is desirable that the system should be gradually introduced of requiring that a candidate, in order to secure a pass, should evince some power besides that of being able to write out bookwork.

12. It is desirable that, when possible, in an examination in geometry, there should be a paper of exercises, including the practical use of instruments, and the solution of riders.

ORDER OF TEACHING THE EARLIER BOOKS.

13. The Committee recommend the following general order in teaching the *theorems* of the first three books, and think that examiners should be requested to recognise this order :—

Book I.
Book III. to 32 inclusive.
Book II.
Book III. 35 to the end.

14. The detailed order must depend largely on some of the definitions, and on the methods of proof admitted in certain propositions. The Committee consider it desirable in the teaching of the subject that Books I. and III. be taken more or less concurrently.

DETAILED SUGGESTIONS.

15. That definitions should not be taught *en bloc* at the beginning of each book, but that each definition should be introduced when required.

BOOK I.

(i) *Theorems.*

16. That 13, 14, 15 be taken first.

17. That a very short proof of 13 should be considered satisfactory.

18. That 7 be omitted.

19. That, when two triangles are congruent, the equal parts should be written *side by side in two columns* :—

e.g. in $\triangle$s ABC, DEF,
$AB = DE$,
$AC = DF$,
and $\angle BAC = \angle EDF$;
$\therefore$ the $\triangle$s are congruent.

20. That 8 be proved by placing the triangles in opposition.

21. That proofs of 24 by 19, which are incomplete, should be amended; but that proof by 20 should be preferred.

22. That 26 be proved by superposition.

23. That, in connection with I. 4, 8, 26, the following proposition be introduced :—

Two right-angled triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are congruent. (This can be proved by placing the triangles in opposition with their equal sides coincident, and applying I. 5 and 26.)

24. That the following propositions be introduced :—

(1) *The locus of points equidistant from two given points is the perpendicular bisector of the line joining the given points.*

(2) *The locus of points equidistant from two given intersecting straight lines is the pair of bisectors of the angles contained by the given lines.*

25. That Playfair's axiom is preferable to Euclid's 12th axiom.

26. That, in dealing with angles connected with parallel lines, triangles, and polygons, illustration by rotation is desirable.

27. That it should be proved (for commensurables) that the area of a parallelogram is measured by the product of the measures of its base and height, and the area of a triangle by half this product. (Cf. § 47.)

(ii) *Constructions.*

28. That 1 be replaced by 22.

29. That 2, 3 be omitted.

30. That, in 45, the figure should first be reduced to a triangle by an application of I. 38.

BOOK II.

31. It is proposed that this book should be taken after III. 32; suggestions with regard to this book will be found below.

BOOK III.

32. That there should be a preliminary discussion of such fundamental properties of the circle as the following:—

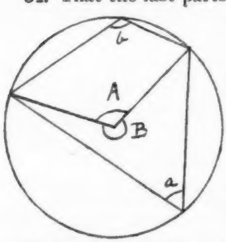
Every diameter is a line of symmetry: *i.e.* if the figure be folded about a diameter, the two portions of the circle coincide: with the corollaries:—

(1) The two ends of every chord drawn perpendicular to the diameter are equidistant from it and from any point in it. Such a pair of points may be called symmetrically opposite points with regard to the diameter, or one may be called the image of the other in the diameter.

(2) The line of centres (*i.e.* the line drawn through the centres) of two circles is a line of symmetry. Hence, if the circles cut in a point off the line of centres, they also cut in the symmetrically opposite point; and, if the circles touch, either internally or externally, the point of contact must be on the line of centres; also, if two circles have the same centre, they either do not meet at all, or coincide entirely.

33. That 2, 4, 5, 6, 10, 11, 12, 13 be omitted.

34. That the last parts of 7 and 8 be omitted.



35. That the 'limit' definition of a tangent be allowed.

36. That 16, 18, 19 be replaced by the proposition, *The tangent at any point of a circle and the radius to the point of contact are at right angles to one another*: with the corollary, *One and only one tangent can be drawn at any point of a circle.*

37. That in 20, 21, 22 the use of angles greater than two right angles be allowed (ambiguity is rendered impossible if the angles are lettered instead of the points).

38. That 23, 24 be omitted.

39. That 26, 27 be stated as one proposition, and be proved by superposition, and that the equality of the sectors be proved as a corollary.

BOOK II.

40. That the following definitions of a *rectangle* and a *square* be accepted:—

A *rectangle* is a parallelogram which has one of its angles a right angle.

A *square* is a rectangle which has two adjacent sides equal.

41. That those proofs are preferable which do not make use of the diagonal.

42. That illustration from Algebra ought to be given where such is possible.

43. That 8, 9, 10 be omitted.

44. That the use of the signs + and − be allowed.

BOOK IV.

45. That all propositions be omitted, as formal propositions, except 2, 3, 4, 5, 10, and that these be taken with earlier books; the rest of the book being treated as exercises in geometrical drawing.

BOOK VI.

46. During the preliminary course of geometry it should be pointed out that drawings of different sizes can present exactly the same appearance, and that, in such drawings, the corresponding lines are proportional and the corresponding angles equal. Practical problems in heights and distances can in this way be solved graphically by quite young pupils, and are found most interesting exercises.

47. The committee suggest that the study of Book VI. should be preceded by

(1) A theory of measurement of lengths of lines and areas of rectangles for cases in which the lines and the sides of the rectangles are commensurable. (Cf. § 27.)

(2) An algebraical treatment of ratio and proportion for commensurables.

48. That an ordinary school course should not be required to include incommensurables;—in other words, that in such a course all magnitudes of the same kind should be treated as commensurable.

49. This limitation would necessarily involve a change of proof in VI. 1 and 33; for VI. 1, either of the following methods of proof might be adopted:—

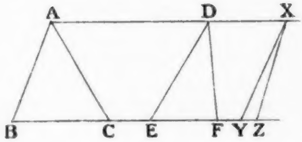
(1) Let ABC, DEF be two $\triangle$s whose bases BC, EF are commensurable and whose heights are equal.

Place the $\triangle$s so that the bases BC, EF are in the same straight line and so that the $\triangle$s are on the same side of the line.

BC, EF being commensurable, have a common measure. Let YZ be a common measure, and let BC and EF contain YZ p times and q times respectively.

Place YZ on $BCEF$; take any point X in AD, and join XY, XZ.

Since the $\triangle$s ABC, DEF have equal heights, AD is parallel to BF (a proof of this should be given if there is no proposition or corollary to refer to).



Hence $\triangle ABC$ is p times the $\triangle XYZ$, and $\triangle DEF$ is q times the $\triangle XYZ$;

$\therefore\ \triangle ABC : \triangle DEF = p : q$
$\qquad = BC : EF,$

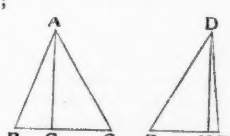
i.e. etc., etc. Q.E.D.

(2) Let ABC, DEF be two $\triangle$s of the same height.

Draw AG, DH perpendicular to BC, EF respectively, then $AG = DH$.

Now the area of a triangle is measured by half the product of the measures of its base and height (see § 27);

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC \,.\, AG}{\frac{1}{2}EF \,.\, DH} = \frac{BC \,.\, AG}{EF \,.\, DH} = \frac{BC}{EF}, \text{ since } AG = DH. \quad \text{Q.E.D.}$$



50. The Committee desire to call attention to the method of proof of an extension of VI. 2 in the A.I.G.T. Geometry.

51. In stating the conditions for the similarity of two triangles ABC, DEF,

$$\frac{BC}{EF}=\frac{CA}{FD}=\frac{AB}{DE}$$

(or $BC : EF = CA : FD = AB : DE$) is preferable to

$$\frac{AB}{BC}=\frac{DE}{EF},\ \frac{BC}{CA}=\frac{EF}{FD},\ \text{and}\ \frac{CA}{AB}=\frac{FD}{DE}.$$

52. The expression *corresponding sides* is preferable to the expression *homologous sides.*

53. In connection with the formal course, as soon as the proposition that equiangular triangles are similar has been proved, the sine, cosine, and tangent can be defined (if this has not been done earlier in the experimental course). In order to make the meanings and importance of these functions sink deeply into the pupil's mind, numerical examples should be given on right-angled triangles (heights and distances); these should be worked with the help of four-figure tables.

54. The expression for the area of a triangle, and of a parallelogram, in terms of two sides and the included angle, may be introduced simultaneously with the propositions concerning areas in Book VI.

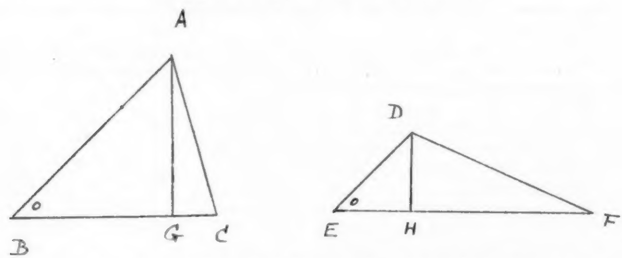
55. In accordance with the spirit of the above proposals, the Committee suggest that the following proposition be adopted:—

If two triangles (or parallelograms) have one angle of the one equal to one angle of the other, their areas are proportional to the areas of the rectangles contained by the sides about the equal angles.

If the $\triangle$s ABC, DEF have $\angle B = \angle E$, their areas are in the ratio $\dfrac{AB \,.\, BC}{DE \,.\, EF}$.

Draw AG, DH perpendicular to BC, EF respectively.

Then $$\frac{\triangle ABC}{\triangle DEF}=\frac{\frac{1}{2}BC\,.\,AG}{\frac{1}{2}EF\,.\,DH}=\frac{BC\,.\,AG}{EF\,.\,DH}.$$



Also, since $\angle B = \angle E$ and $\angle G = \angle H$, the $\triangle$s ABG, DEH are similar;

$$\therefore\ \frac{AG}{DH}=\frac{AB}{DE};$$

$$\therefore\ \frac{\triangle ABC}{\triangle DEF}=\frac{BC\,.\,AB}{EF\,.\,DE}.$$ Q.E.D.

56. 19 follows immediately in the form, *The areas of similar triangles are proportional to the squares on corresponding sides.*

57. 20 can be deduced as in Euclid.

58. 22 follows at once.

59. 14, 15, 16, 17, 21, 23, 24, 26, 27, 28, 29, 32 should be omitted.

60. All statements of ratio may be made in fractional form, and the sign = used instead of :: (as has been done above).

61. In the ordinary school course, reciprocal proportion should be dropped and compounding replaced by multiplying.

SOME APPLICATIONS OF THE THEORY OF ASSEMBLAGES.

BY ARNOLD EMCH.

(*Presented to the Am. Math. Soc.*, 28*th December*, 1901.)

1. In order to be able to apply the theory of real numbers to measurable quantities,* in particular to linear magnitudes, it is convenient to make use of the theory of groups (displacements). Thus, if $B, C, D, \ldots, Y$ are successive points on a continuous line between the extremities A and Z,

$$AB+BC+CD+\ldots+YZ=AZ,$$

where $AB, BC, \ldots$ are arcs or consecutive displacements on the given line. Assuming $AB=u$ as a unit-displacement, then u may be repeated a certain number of times, say a times, so that

$$au<AZ<(a+1)u.$$

Designating the remainder YZ by u_1,

$$AZ-au=u_1.$$

With u_1 as a second unit-displacement operate on the original unit u in exactly the same manner as before on AZ. If b is the corresponding positive integer,

$$bu_1<u<(b+1)u_1.$$

Continuing this process until it closes, or else indefinitely, the original displacement AZ measured by the unit-displacement $u=1$ becomes

$$AZ=a+\frac{1}{b+}\frac{1}{c+}\frac{1}{d+}\ldots,$$

and this is called the length of the line AZ. If the process closes with a finite number of terms, AZ is a rational number, if it does not close,

$$AZ=\lim\left(a+\frac{1}{b+\ldots}\right)$$

is an irrational number.

This so-called *Euclidian algorithm* may also be applied to measure the length of a closed curve. From the form of the algorithm it is apparent that the length of every finite continuous line is represented by some real positive number, and conversely, every positive real number may be represented by the length of some continuous line. The theory may, of course, be immediately extended to negative real numbers. For, from $AB+BA=0$ it follows $BA=-AB$, and from this standpoint negative and positive real quantities do not present essential differences in their treatment.

2. I shall now consider a circle† whose perimeter may be any real positive number. To prove this, let S be any real number, then the perimeter of a circle will be S if its radius is $\frac{S}{2\pi}$. Choose now any unit-displacement $u<S$ on the circle and suppose that n_1 is a positive integer, so that

$$n_1u<S<(n_1+1)u.$$

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* See in this connection,

Lagrange: *Lectures on Elementary Mathematics*, p. 3; translation by Th. J. Carnack.

Dedekind: *Essays on the Theory of Numbers*, pp. 1-27; translation by Prof. Beeman.

E. Borel, *Leçons sur la Théorie des Fonctions*, vol. i., p. 111.

† Any other closed curve would do; the circle has been chosen on account of its simplicity.

Put the remainder $S-n_1u=u_1$, then $u_1<u$. With u_1 as a new unit repeat the previous process, so that if n_2 is another positive integer

$$n_2u_1<S<(n_2+1)u_1.$$

Clearly $n_2>n_1$. Put $S-n_2u_1=u_2$, where $u_2<u_1$, and so forth. In this manner we get

$$\begin{aligned}S&=n_1u+u_1\\ S&=n_2u_1+u_2\\ S&=n_3u_2+u_3\\ &\dots\dots\dots\dots\\ S&=n_ku_{k-1}+u_k,\\ &\dots\dots\dots\dots\end{aligned}$$

where $$u>u_1>u_2>\dots>u_k>\dots,$$

and $$n_1<n_2<n_3<\dots<n_{k+1}<\dots.$$

Eliminating $u_1, u_2, u_3, \dots, u_{k-1}$ between the first k equations we obtain

$$S=\frac{u_k+(-1)^{k+1}n_1n_2n_3\dots n_ku}{1-n_k+n_kn_{k-1}-\dots+(-1)^{k+1}n_2n_3\dots n_k},$$

or $$S=\frac{u+(-1)^{k+1}\dfrac{u_k}{n_1n_2n_3\dots n_k}}{\dfrac{1}{n_1}-\dfrac{1}{n_1n_2}+\dfrac{1}{n_1n_2n_3}-\dots+(-1)^{k+1}\dfrac{1}{n_1n_2n_3\dots n_k}}.$$

If this algorithm is limited, then S is rational, if it is unlimited, $\lim(n_k)=\infty$, $\lim(u_k)=0$, and

(I) $$S=\frac{u}{\dfrac{1}{n_1}-\dfrac{1}{n_1n_2}+\dfrac{1}{n_1n_2n_3}-\dots}.$$

If $u=1$ and $S>1$ we have

$$S=\frac{1}{\dfrac{1}{n_1}-\dfrac{1}{n_1n_2}+\dfrac{1}{n_1n_2n_3}-\dots}.$$

If S is a real number, $\frac{1}{S}$ is also a real number, so that for $u=1$ and $S<1$ we have

(II) $$S=\frac{1}{n_1}-\frac{1}{n_1n_2}+\frac{1}{n_1n_2n_3}-\dots.\text{*}$$

Every real number may thus be represented by I. *or* II.

As examples I mention

$$e=\frac{1}{\dfrac{1}{2}-\dfrac{1}{2.3}+\dfrac{1}{2.3.4}-\dfrac{1}{2.3.4.5}+\dots},$$

$$\frac{e}{e-1}=\frac{1}{\dfrac{1}{1}-\dfrac{1}{1.2}+\dfrac{1}{1.2.3}-\dfrac{1}{1.2.3.4}+\dots}.$$

Applying the algorithm to the Ludolphian number π, it is found

$$\pi=\frac{1}{\dfrac{1}{3}-\dfrac{1}{3.22}+\dfrac{1}{3.22.118}-\dfrac{1}{3.22.118.390}+\dots}.$$

In this infinite expression it would be very interesting if the law of its general term could be established.

* See an interesting note of E. Maillet, *Comptes Rendus*, vol. cxxxiii., No. 20, pp. 782-784. "Sur les équations différentielles rationelles."

3. Geometrically, if the unit-displacement u is repeated indefinitely, two cases may occur. The polygon formed by the extremities of all displacements closes, or it does not close. It closes when $\frac{S}{u}$ is a rational number, for if $\frac{S}{u}=\frac{m}{n}$, where m and n are positive integers with no common factors, $nS=mu$, *i.e.* if u is repeated m times, the circle is described n times. Hence, after m displacements u, we arrive at A, the starting point. If $\frac{S}{u}$ is irrational we easily conclude from our algorithm that after

$$n_1+n_1n_2+n_1n_2n_3+\ldots+n_1n_2n_3\ldots n_k$$

displacements u we arrive at a point A_k so that

$$AA_k=n_ku_{k-1}\,;\ S-n_ku_{k-1}=u_k<u_{k-1}.$$

As $$u>u_1>u_2>\ldots>u_k>\ldots,$$

we can make u_k as small as we please by making k sufficiently great.

Hence we can make $$A_kA<\delta,$$

δ being arbitrarily small.

The points of the infinite polygon come therefore arbitrarily close to the starting-point A. Now, any point of the polygon may be chosen as a starting-point and the same conclusion holds for this point. Hence the theorem: *The vertices of a polygon of an infinite number of equal sides inscribed into a circle form an enumerable system of points which are everywhere dense.*

UNIVERSITY OF COLORADO,
December, 1901.

NOTES ON CONICS IN AREALS.

In a recent number of the *Proceedings of the Cambridge Philosophical Society* (vol. x., 1900, p. 358), I have considered at some length the problem of reducing conics and quadrics to principal axes. Since that note was published I have succeeded in effecting the reduction by means of more familiar methods; the recent publication of Mr. Muggeridge's notes on Areals seems to lead up to mine.

The basis of my investigation is the following theorem, which should be familiar to most readers: If a quadratic form S is transformed to T by means of any linear substitution (whose determinant is M), then the determinant of T is M^2 times that of S; or, symbolically,

$$|T|=|S|\times M^2.$$

(Salmon, *Conics*, 1879, note, p. 335.)

Suppose now that we wish to pass from areal coordinates $(x,\ y,\ z)$ to rectangular $(\xi,\ \eta)$; and let $(\xi_1,\ \eta_1)$, $(\xi_2,\ \eta_2)$, $(\xi_3,\ \eta_3)$ be the rectangular coordinates of the angular points of the triangle of reference. Then we have the substitution

$$\xi=x\xi_1+y\xi_2+z\xi_3,\quad \eta=x\eta_1+y\eta_2+z\eta_3,\quad 1=x+y+z.$$

Thus the determinant of the substitution is

$$\begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ \eta_1, & \eta_2, & \eta_3 \\ 1, & 1, & 1 \end{vmatrix}=2\Delta,$$

if Δ is the area of the triangle of reference.

It follows from the general result that, if we have

$$(p, q, r, p', q', r' \between \xi, \eta, 1)^2 = (u, v, w, u', v', w' \between x, y, z)^2,$$

then

$$\begin{vmatrix} p, & r', & q' \\ r', & q, & p' \\ q', & p', & r \end{vmatrix} (2\Delta)^2 = \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}.$$

Following Mr. Muggeridge's notation, we shall, in general, denote the determinant of $(u, v, w, u', v', w' \between x, y, z)^2$ by the symbol H; and the minors of $u, v, \ldots$ in H, by $U, V \ldots$.

We next observe that the expression $-(a^2yz+b^2zx+c^2xy)$ represents the square of the tangent from (x, y, z) to the circumcircle of the triangle of reference, and so takes the form $\xi^2+\eta^2+2g\xi+2f\eta+c$ when transformed to rectangular coordinates.* It follows that the two forms,

$$(u, v, w, u', v', w' \between x, y, z)^2 + \lambda(a^2yz+b^2zx+c^2xy),$$

$$(p, q, r, p', q', r' \between \xi, \eta, 1)^2 - \lambda(\xi^2+\eta^2+2g\xi+2f\eta+c),$$

represent the same form, which we shall denote by S_1. The values of λ for which the conic $S_1=0$ is a parabola are determined by

$$\begin{vmatrix} u, & w'+\frac{1}{2}\lambda c^2, & v'+\frac{1}{2}\lambda b^2, & 1 \\ w'+\frac{1}{2}\lambda c^2, & v, & u'+\frac{1}{2}\lambda a^2, & 1 \\ v'+\frac{1}{2}\lambda b^2, & u'+\frac{1}{2}\lambda a^2, & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

and also by

$$\begin{vmatrix} p-\lambda, & q \\ q, & r-\lambda \end{vmatrix} = 0.$$

Expanding these, we have that the two equations in λ,

$$4\Delta^2\lambda^2 - I\lambda + K = 0,$$

$$\lambda^2 - (p+r)\lambda + (pr-q^2) = 0,$$

have the same roots; where, for brevity, we write

$$I = a^2u + b^2v + c^2w - 2bcu'\cos A - 2cav'\cos B - 2abw'\cos C,$$

$$K = U + V + W + 2U' + 2V' + 2W'.$$

[It should be remarked that I have slightly altered Mr. Muggeridge's notation here, as this is his $-K$.]

Suppose, in the first place, that our conic can be reduced to the form $\alpha\xi^2+\beta\eta^2+\gamma=0$. Then, from the two results just obtained, it follows that

$$(\alpha\beta\gamma)(4\Delta^2) = H,$$

and that α, β are the roots of $4\Delta^2\lambda^2 - I\lambda + K = 0$.

Hence $\gamma = H/4\Delta^2\alpha\beta = H/K$; and the semi-axes (ρ) are given by

$$\rho^2 = -\gamma/\alpha, \quad -\gamma/\beta,$$

so put $\rho^2 = -\gamma/\lambda = -H/K\lambda$, and we find the quadratic

$$K^3\rho^4 + HIK\rho^2 + 4\Delta^2H^2 = 0$$

as obtained by Mr. Muggeridge.

If $H=0$, $K \neq 0$, the form just obtained holds good, with $\gamma=0$. We have here *two intersecting straight lines*; the angle between them is

$$2\tan^{-1}(-\alpha/\beta)^{\frac{1}{2}} = \tan^{-1}[4\Delta(-K)^{\frac{1}{2}}/I],$$

for α, β are roots of $\quad 4\Delta^2\lambda^2 - I\lambda + K = 0.$

* This can also be verified by the substitution for $\xi, \eta, 1$, in terms of x, y, z.

But if $K=0$, $H\neq 0$, the equation for λ has one zero root, and we have to take as the reduced form $a\xi^2+2\gamma\eta=0$. Then

$$-a\gamma^2(4\Delta^2)=H,$$

and $\lambda=a$ is the non-zero root of $4\Delta^2\lambda^2-I\lambda=0$. Thus, if l is the semi-latus-rectum of the parabola, we have

$$l^2=\gamma^2/a^2=-16\Delta^4H/I^3,$$

as given by Mr. Muggeridge.

If $H=0$, $K=0$, the method fails, and we have to use the two expressions for the determinants of S_1; the reduced form $a\xi^2+\gamma=0$ is possible, and so we get

$$(-a\gamma\lambda+\ldots)4\Delta^2=(a^2U'+b^2V'+c^2W')\lambda+\ldots,$$

i.e. $\qquad a\gamma=-(a^2U'+b^2V'+c^2W')/4\Delta^2$;

and, as before, $\qquad a=I/4\Delta^2$,

so that $\qquad -\gamma/a=(a^2U'+b^2V'+c^2W')4\Delta^2/I^2$.

The conic is here *a pair of parallel straight lines*, at a distance $2(-\gamma/a)^{\frac{1}{2}}$ apart.

If, in the last case, $\gamma=0$, we must have $L=a^2U'+b^2V'+c^2W'=0$, which, in conjunction with $H=0$, $K=0$, leads to the condition that *all* the first minors of H must vanish; here the conic is *a pair of coincident straight lines.*

We may have $H=0$, $I=0$, $K=0$, but $L\neq 0$, then the reduced form is $2\gamma\eta$, where $\gamma^2=-L/4\Delta^2$; which, as in the last case, is obtained by equating the two expressions for the determinant of S_1. The conic consists of *a straight line and the line at infinity.*

If $I=0$, and *all* the first minors of H vanish, the conic either represents the line at infinity squared, or else is identically zero.

Classifying the cases, we have

(i) $H\neq 0$, $K\neq 0$, *a central conic.*

(ii) $H\neq 0$, $K=0$, *a parabola.*

(iii) $H=0$, $K\neq 0$, *intersecting straight lines.*

(iv) $H=0$, $K=0$,
- (α) $I\neq 0$, $L\neq 0$, *parallel lines*;
- (β) $I\neq 0$, $L=0$, *coincident lines*;
- (γ) $I=0$, $L\neq 0$, *a line and the line* ∞;
- (δ) $I=0$, $L=0$, *the line* ∞ (*twice*).

If we attempt to form the line-equations corresponding to these point-equations, we see that, in every case where $H=0$, the line-equation does not give complete information, and in some cases [viz. (β), (δ)] it does not exist at all.

Line Equations.

Just as the point-equation of some conics has no line-equation, or an equation which does not, of itself, completely specify the conic; so it may happen that the point-equation corresponding to a given line-equation does not give full information as to the conic. We must, therefore, consider a means of discussing conics starting from the line equation.

Take the equation $(u_0, v_0, w_0, u'_0, v'_0, w'_0 \between l, m, n)^2=0$ as the line-equation of a conic; if there is a corresponding point-equation

$$(u, v, w, u', v', w' \between x, y, z)^2=0,$$

we shall have, as usual,

$$\frac{u_0}{U}=\frac{v_0}{V}=\frac{w_0}{W}=\frac{u'_0}{U'}=\frac{v'_0}{V'}=\frac{w'_0}{W'}.$$

Suppose that the line $lx+my+nz=0$ becomes $\lambda\xi+\mu\eta+\nu=0$, and further that

$$lx+my+nz=\lambda\xi+\mu\eta+\nu.$$

Then $\quad l=\lambda\xi_1+\mu\eta_1+\nu,\quad m=\lambda\xi_2+\mu\eta_2+\nu,\quad n=\lambda\xi_3+\mu\eta_3+\nu,$

and the determinant of this substitution is

$$\frac{\partial(l,m,n)}{\partial(\lambda,\mu,\nu)}=\begin{vmatrix}\xi_1, & \eta_1, & 1\\ \xi_2, & \eta_2, & 1\\ \xi_3, & \eta_3, & 1\end{vmatrix}=2\Delta.$$

Solving for λ, μ in terms of l, m, n, we have

$$\lambda=\frac{1}{2\Delta}[l(\eta_2-\eta_3)+m(\eta_3-\eta_1)+n(\eta_1-\eta_2)],$$

$$\mu=\frac{-1}{2\Delta}[l(\xi_2-\xi_3)+m(\xi_3-\xi_1)+n(\xi_1-\xi_2)].$$

Hence $\lambda^2+\mu^2=\dfrac{1}{4\Delta^2}[a^2l^2+b^2m^2+c^2n^2-2bcmn\cos A-2canl\cos B-2ablm\cos C]$,

which can also be verified by comparing the expressions for the perpendicular on the line.

Thus, if the given equation is equivalent to

$$(p_0, q_0, r_0, p'_0, q'_0, r'_0 \between \lambda, \mu, \nu)^2,$$

by comparing the determinants of the equivalent forms,

$$(u_0, v_0, w_0, u'_0, v'_0, w'_0 \between l, m, n)^2-\phi(a^2l^2+\ldots-2bcmn\cos A-\ldots)$$

and $\quad (p_0, q_0, r_0, p'_0, q'_0, r'_0 \between \lambda, \mu, 1)^2-\theta(\lambda^2+\mu^2),$

we find that

$$\begin{vmatrix}p_0-\theta, & r'_0 & q'_0\\ r'_0, & q_0-\theta, & p'_0\\ q'_0, & p'_0, & r_0\end{vmatrix}=4\Delta^2\begin{vmatrix}u_0-a^2\phi, & w'_0+ab\phi\cos C, & v'_0+ca\phi\cos B\\ w'_0+ab\phi\cos C, & v_0-b^2\phi, & u'_0+bc\phi\cos A\\ v'_0+ca\phi\cos B, & u'_0+bc\phi\cos A, & w_0-c^2\phi\end{vmatrix},$$

where we write for brevity $\qquad \phi=\theta/4\Delta^2.$

Expanding, we have

$$\begin{vmatrix}p_0-\theta, & r'_0, & q'_0\\ r'_0, & q_0-\theta, & p'_0\\ q'_0, & p'_0, & r_0\end{vmatrix}=4\Delta^2(H_0-I_0\phi+4K_0\Delta^2\phi^2)$$
$$=K_0\theta^2-I_0\theta+4\Delta^2H_0,$$

where $\quad H_0=\begin{vmatrix}u_0, & w'_0, & v'_0\\ w'_0, & v_0, & u'_0\\ v'_0, & u'_0, & w_0\end{vmatrix},$

$$K_0=u_0+v_0+w_0+2u'_0+2v'_0+2w'_0,$$

$$I_0=a^2U_0+b^2V_0+c^2W_0-2bcU'_0\cos A-2caV'_0\cos B-2abW'_0\cos C,$$

U_0, V_0, ... being the minors of u_0, v_0, ... in H_0.

It should be observed that, if $H_0 \neq 0$, we can write $u_0=U$, $v_0=V$, ... and then, comparing with the former notation,

$$H_0=H^2,\quad I_0=HI,\quad K_0=K.$$

In the case of a non-degenerate central conic, we can write

$$p'_0=0,\quad q'_0=0,\quad r'_0=0,$$

and then $\qquad r_0(p_0-\theta)(q_0-\theta)=K_0\theta^2-I_0\theta+4\Delta^2H_0.$

The semi-axes are given by

$$\rho^2 = -p_0/r_0, \quad -q_0/r_0,$$

so write

$$\rho^2 = -\theta/r_0 = -\theta/K_0,$$

and the semi-axes are seen to be roots of

$$K_0^3\rho^4 + I_0K_0\rho^2 + 4\Delta^2 H_0 = 0,$$

which agrees with the result found from the point-equation, if we express H_0, I_0, K_0 in terms of H, I, K. The result also agrees with Salmon's (*Conics*, Art. 382 and (ii), p. 392).

This result still holds if H_0 or I_0 is zero, so long as $K_0 \neq 0$; and if $H_0=0$, $I_0 \neq 0$, the conic degenerates to a *pair of points*, at a distance apart

$$2(-p_0/r_0)^{\frac{1}{2}} = 2(-I_0/K_0^3)^{\frac{1}{2}}.$$

If $H_0=0$ and $I_0=0$, the *two points are coincident.*

But if $K_0=0$, we must have $r_0=0$, and the last form breaks down. We can, however, assume $q_0=0$, $q'_0=0$, $r'_0=0$.

Then

$$-p_0'^2(p_0-\theta) = K_0\theta^2 - I_0\theta + 4\Delta^2 H_0,$$

so that

$$p_0'^2 = -I_0, \quad p_0 p_0'^2 = -4\Delta^2 H_0,$$

and the form $p_0\lambda^2 + 2p'_0\mu\nu = 0$ represents *a parabola* of semi-latus-rectum $p_0/p'_0 = 4\Delta^2 H_0/(-I_0)^{\frac{3}{2}}$, agreeing with the result obtained from the point-equation.

But, if $H_0=0$, $I_0 \neq 0$, we have $p_0=0$, and the reduced equation is $2p'_0\mu\nu=0$, representing *two points* (*one at infinity*).

Finally, if $H_0=0$, $I_0=0$, $K_0=0$, the determinant vanishes identically, and so the reduced form is $p_0\lambda^2+q_0\mu^2=0$, which represents *two points at infinity.* In this case, none of the methods already used can be effective; we find that the conic can be expressed in terms of the two differences $(l-m)$, $(l-n)$. So the simplest method is to put, say, $l=0$, and consider

$$(p_0-\theta)\lambda^2 + (q_0-\theta)\mu^2 = (v_0 - b^2\phi)m^2 + (w_0 - c^2\phi)n^2 + 2(u'_0 + bc\phi\cos A)mn,$$

where

$$\phi = \theta/4\Delta^2.$$

Thus

$$(p_0-\theta)(q_0-\theta) = M^2 \begin{vmatrix} v_0 - b^2\phi, & u'_0 + bc\phi\cos A \\ u'_0 + bc\phi\cos A, & w_0 - c^2\phi \end{vmatrix}$$

$$= M^2\left[U_0 - \phi(b^2w_0 + c^2v_0 + 2bcu'_0\cos A) + 4\phi^2\Delta^2\right],$$

where M is the determinant of the substitution for λ, μ in terms of m, n.

Now the conditions $H_0=0$, $I_0=0$, $K_0=0$ are found to give

$$u_0 + v'_0 + w'_0 = 0, \quad u'_0 + v_0 + w'_0 = 0, \quad u'_0 + v'_0 + w_0 = 0;$$

so that

$$U_0 = v'_0w'_0 + w'_0u'_0 + u'_0v'_0,$$

$$b^2w_0 + c^2v_0 + 2bcu'_0\cos A = -(a^2u'_0 + b^2v'_0 + c^2w'_0).$$

Thus p_0, q_0 are the roots of the equation in θ,

$$\theta^2 + \theta(a^2u'_0 + b^2v'_0 + c^2w'_0) + 4\Delta^2(v'_0w'_0 + w'_0u'_0 + u'_0v'_0) = 0,$$

but we really need only the ratio of these roots.

Most, if not all, of the above results will be found in Prof. S. Gundelfinger's *Lectures on Conics* (Leipzig, 1895), though his methods are different from mine; but I do not think that the question has been treated, *completely*, by any English writer, except in my paper already quoted.

St. John's College, Cambridge,
9th October, 1901.

T. J. I'A. BROMWICH.

CORRESPONDENCE.

To the Editor of the "MATHEMATICAL GAZETTE."

SIR,—Many thanks for your courtesy in sending me the draft report of the Committe on Geometry. I venture to say that it seems to me to be, on the whole, on sound lines. It would be presumptuous on my part to pass judgment on the details of any such scheme; as, for the last twenty-five years, I have had no direct connection with scholastic work; I have been employed only in cases where school methods had failed to convey any comprehension of mathematical procedure. My chief work, however, has consisted in investigating the causes of such failures with the help of medical practitioners, who are studying the laws of normal mental action by the aid of the formula known as "Boole's Equation," and the mathematical analysis of normal thought-sequence. It is gratifying to find that educational authorities are inviting the attention of the public to methods of teaching which store mental power and nerve-stamina, instead of scattering them at random for the mere purpose of producing a showy and false appearance of precocious knowledge.—Yours respectfully, MARY EVEREST BOOLE.

SOLUTION.

391. [K. 20. e.] *In any triangle prove that*

$$(a+b-2c)^2\sec^2\frac{C}{2}+(a-b)^2\operatorname{cosec}^2\frac{C}{2}=(b+c-2a)^2\sec^2\frac{A}{2}+(b-c)^2\operatorname{cosec}^2\frac{A}{2}$$

$$=(c+a-2b)^2\sec^2\frac{B}{2}+(c-a)^2\operatorname{cosec}^2\frac{B}{2},$$

and interpret geometrically. E. N. BARISIEN.

Solution by C. E. YOUNGMAN.

Suppose $a<b<c$. On BC, CA, AB, take $BD=c$, $CE=a$, $AF=b$: and again on CB, BA, AC, take $CD'=b$, $BF'=a$, $AE'=c$. This makes $CD=c-a$, $CE'=c-b$, and so on. Project DE' on the bisectors of the angle C; one projection is $(CD-CE')\cos\frac{1}{2}C$, and the other $(CD+CE')\sin\frac{1}{2}C$;

$$\therefore\ DE'^2=(a+b-2c)\sin^2\tfrac{1}{2}C+(a-b)^2\cos^2\tfrac{1}{2}C;$$

$\therefore$ the first of the given expressions $=DE'^2\sec^2\frac{1}{2}C\operatorname{cosec}^2\frac{1}{2}C$

$=4DE'^2\operatorname{cosec}^2C=$ sq. on twice the diameter of the circle CDE'.

Hence the assertion is that the circles CDE', AEF', BFD' are equal; or that

$$DE':EF':FD'=\sin C:\sin A:\sin B=AB:BC:CA.$$

To prove this, let $E'F$ meet BC at L. We have $BE'\parallel CF$;

$$\therefore\ E'L:LF=BL:LC=BE':CF=AB:AF=c:b$$

$$=BD:CD'=BD-BL:CD'-CL$$

$$=DL:LD';$$

$$\therefore\ DE'\parallel FD';\quad\text{and}\quad DE':FD'=c:b.$$

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